Introduction to barycentric geometry with applications.

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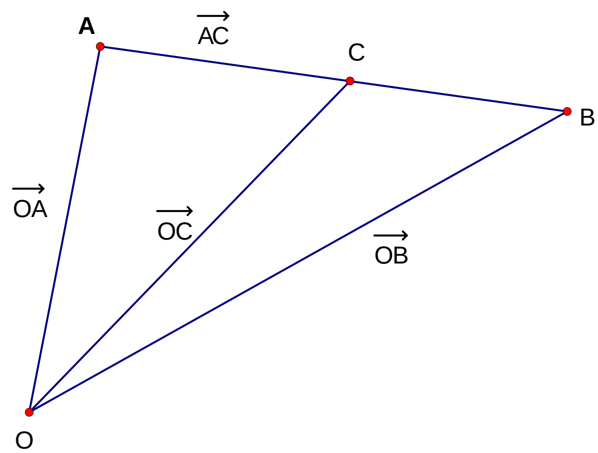
Some preliminary facts.

First recall that any two non collinear vectors $\overrightarrow{OA}, \overrightarrow{OB}$ create a basis on the plane with origin O, that is for any vector $\overrightarrow{OC}$ there are unique $p, q \in \mathbb{R}$ such that $\overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OB}$ and we saying that pair (p, q) is coordinates of $\overrightarrow{OC}$ in the basis $\left(\overrightarrow{OA}, \overrightarrow{OB}\right)$ and $\overrightarrow{OC}$ is linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with coefficients p and q.Also note that point C belong to the segment AB iff $\overrightarrow{OC}$ is linear combination of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ with non negative coefficients p and such that $p + q = 1$.(in that case we saying that $\overrightarrow{OC}$ is convex combination of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ or that segment AB is convex combination of his ends).



Indeed let C belong to the segment AB. If $C \in \{A, B\}$ then $\overrightarrow{AC} = k\overrightarrow{AB}$,where $k \in \{0, 1\}$.If $C \notin \{A, B\}$ then $\overrightarrow{AC}$ is collinear with $\overrightarrow{AB}$ and directed as $\overrightarrow{AB}$,that is $\overrightarrow{AC} = k\overrightarrow{AB}$ for some positive k.Hence, $\left\|\overrightarrow{AC}\right\| = \left\|k\overrightarrow{AB}\right\| = k\left\|\overrightarrow{AB}\right\| \iff k = \dfrac{\left\|\overrightarrow{AC}\right\|}{\left\|\overrightarrow{AB}\right\|} < 1.$

Thus, if C belong to the segment AB then $\overrightarrow{AC} = k\overrightarrow{AB}$ with $k \in [0, 1]$ and since $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA}$, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$ then $\overrightarrow{AC} = k\overrightarrow{AB} \iff \overrightarrow{OC} - \overrightarrow{OA} = k\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \iff \overrightarrow{OC} = k\overrightarrow{OB} - k\overrightarrow{OA} + \overrightarrow{OA} \iff$

$\overrightarrow{OC} = (1-k)\overrightarrow{OA} + k\overrightarrow{OA} \iff \overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OA}$, where $p := 1-k, q := k$,that is $p, q \geq 0$ and $p + q = 1$.

Opposite, let $\overrightarrow{OC} = p\overrightarrow{OA} + q\overrightarrow{OB}$,where $p + q = 1$ and $p, q \geq 0$.Then, by reversing transformation above we obtain $\overrightarrow{AC} = q\overrightarrow{AB}, q \in [0, 1]$ .and since $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} - q\overrightarrow{AB} = (1-q)\overrightarrow{AB}$ we obtain

$\left\|\overrightarrow{AC}\right\| = q\left\|\overrightarrow{AB}\right\|, \left\|\overrightarrow{CB}\right\| = (1-q)\left\|\overrightarrow{AB}\right\|$. Therefore, $\left\|\overrightarrow{AB}\right\| = \left\|\overrightarrow{AC}\right\| + \left\|\overrightarrow{CB}\right\| \iff C$ belong to the segment AB.

(Another variant:

Let $\mathbf{a} := \overrightarrow{OA}, \mathbf{b} := \overrightarrow{OB}$ and $\mathbf{c} := \overrightarrow{OC}$.Note that $C \in AB$ iff $\mathbf{c}-\mathbf{a}$ is collinear to $\mathbf{b}-\mathbf{a}$, that is $\mathbf{c}-\mathbf{a} = k(\mathbf{b}-\mathbf{a})$ for some real k and $|AC| + |CB| = |AB|$, that is $\|\mathbf{c}-\mathbf{a}\| + \|\mathbf{b}-\mathbf{c}\| = \|\mathbf{b}-\mathbf{a}\|$ .Thus,

$$C \in AB \iff \begin{cases} \mathbf{c}-\mathbf{a} = k(\mathbf{b}-\mathbf{a}) \\ \|\mathbf{c}-\mathbf{a}\| + \|\mathbf{b}-\mathbf{c}\| = \|\mathbf{b}-\mathbf{a}\| \end{cases}$$

.

Since

$$\mathbf{b}-\mathbf{c} = \mathbf{b}-\mathbf{a}-(\mathbf{c}-\mathbf{a}) = \mathbf{b}-\mathbf{a}-k(\mathbf{b}-\mathbf{a}) = (1-k)(\mathbf{b}-\mathbf{a})$$

then

$$\|\mathbf{c}-\mathbf{a}\| + \|\mathbf{b}-\mathbf{c}\| = \|\mathbf{b}-\mathbf{a}\| \iff \|k(\mathbf{b}-\mathbf{a})\| + \|(1-k)(\mathbf{b}-\mathbf{a})\| = \|\mathbf{b}-\mathbf{a}\| \iff$$

$$|k|\,\|(\mathbf{b}-\mathbf{a})\| + |(1-k)|\,\|(\mathbf{b}-\mathbf{a})\| = \|\mathbf{b}-\mathbf{a}\| \iff |k| + |(1-k)| = 1 \iff 0 \le k \le 1.$$

Hence, $C \in AB \iff \mathbf{c}-\mathbf{a} = k(\mathbf{b}-\mathbf{a}) \iff \mathbf{c} = \mathbf{a}(1-k) + k\mathbf{b}$, where $k \in [0,1]$.

Barycentric coordinates.

Let A, B, C be vertices of non-degenerate triangle. Then, since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ non-colinear, then for each point P on plain we have unique representation $\overrightarrow{AP} = k\overrightarrow{AB} + l\overrightarrow{AC}$,where $k, l \in \mathbb{R}$. Let O be a any point fixed on the plain. Then since $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}, \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$, $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ we obtain $\overrightarrow{AO} + \overrightarrow{OP} = k\left(\overrightarrow{AO} + \overrightarrow{OB}\right) + l\left(\overrightarrow{AO} + \overrightarrow{OC}\right) \iff \overrightarrow{OP} = (1-k-l)\overrightarrow{OA} + k\overrightarrow{OB} + l\overrightarrow{OC}$.Denote $p_a := 1-k-l, p_b := k, p_c := l$, then $p_a + p_b + p_c = 1$ and $\overrightarrow{OP} = p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC}$.

Suppose we have another such representation $\overrightarrow{OP} = q_a\overrightarrow{OA} + q_b\overrightarrow{OB} + q_c\overrightarrow{OC}$ with $q_a + q_b + q_c = 1$, then $\overrightarrow{AP} = p_b\overrightarrow{AB} + p_c\overrightarrow{AC} = q_b\overrightarrow{AB} + q_c\overrightarrow{AC} \implies p_b = q_b, p_c = q_c \implies p_a = q_a$.

Since for each point P we have unique ordered triple of real numbers (p_a, p_b, p_c) which satisfy to condition $p_a + p_b + p_c = 1$ and since any such ordered triple determine some point on plain, then will call such triples barycentric coordinates of point P with respect to triangle ΔABC, because in reality barycentric coordinates independent from origin O. Indeed let O_1 another origin, then

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP} = (p_a + p_b + p_c)\overrightarrow{O_1O} + p_a\overrightarrow{OA} + p_b\overrightarrow{OB} + p_c\overrightarrow{OC} =$$

$$p_a\left(\overrightarrow{O_1O} + \overrightarrow{OA}\right) + p_b\left(\overrightarrow{O_1O} + \overrightarrow{OB}\right) + p_c\left(\overrightarrow{O_1O} + \overrightarrow{OC}\right) = p_a\overrightarrow{O_1A} + p_b\overrightarrow{O_1B} + \overrightarrow{O} + p_c\overrightarrow{O_1C}$$

.

If $p_a, p_b, p_c > 0$ then P is interior point of triangle and in that case we have clear geometric interpretation of numbers p_a, p_b, p_c.Really,since $\overrightarrow{OP} = p_a\overrightarrow{OA} + (p_b + p_c)\left(\dfrac{p_b}{p_b+p_c}\overrightarrow{OB} + \dfrac{p_c}{p_b+p_c}\overrightarrow{OC}\right)$ then linear combination $\dfrac{p_b}{p_b+p_c}\overrightarrow{OB} + \dfrac{p_c}{p_b+p_c}\overrightarrow{OC}$ determine some point A_1 on the segment BC,such that

$$\overrightarrow{OA_1} = \frac{p_b}{p_b+p_c}\overrightarrow{OB} + \frac{p_c}{p_b+p_c}\overrightarrow{OC}\ and \overrightarrow{OP} = p_a\overrightarrow{OA} + (p_a + p_b)\overrightarrow{OA_1}.$$

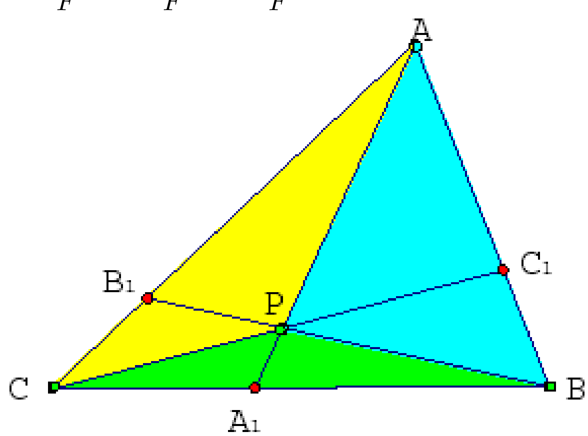
In particularly, $\overrightarrow{AP} = (p_b + p_c)\overrightarrow{OA_1}$.So, P belong to the segment $AA_1$and divide it in the ratio $AP \div PA_1 = (p_b + p_c) \div p_a$.

By the same way we obtain points B_1, C_1 on CA, AB, respectively, and

$$BP \div PB_1 = (p_c + p_a) \div p_b, CP \div PC_1 = (p_a + p_b) \div p_c.$$

Denote $F_a := [PBC]$, $F_b := [PCA], F_c := [PAB]$, $F := [ABC]$ then $p_c \div p_a = AB_1 \div CB_1 = F_c \div F_a, p_a \div p_b = BC_1 \div AC_1 = F_a \div F_b$,$p_b \div p_c =$

$BC_1 \div AC_1 = F_b \div F_c$. So, $p_a \div p_b \div p_c = F_a \div F_b \div F_c$
and $p_a = \dfrac{F_a}{F}, p_b = \dfrac{F_b}{F}, p_c = \dfrac{F_c}{F}$.



Application 1. Barycentric coordinates of some triangle centres.

Problem 1.

Find barycentric coordinates of the following Triangle centers:

a) Centroid G (the point of concurrency of the medians);

b) Incenter I (the point of concurrency of the interior angle bisectors);

c) Orthocenter H of an acute triangle (the point of concurrency of the altitudes);

d) Circumcenter O.

Solution.

a) Since for $P = G$ we have $F_a = F_b = F_c$ then $(p_a, p_b, p_c) = (1/3, 1/3, 1/3)$ is barycentric coordinates of centroid G.

b) Since for $P = I$ we have $\dfrac{F_c}{F_b} = \dfrac{BA_1}{A_1C} = \dfrac{c}{b}, \dfrac{F_a}{F_b} = \dfrac{BC_1}{C_1A} = \dfrac{a}{b}$ then

$F_a \div F_b \div F_c = a \div b \div c$ and, therefore, $(p_a, p_b, p_c) = \dfrac{1}{a+b+c}(a, b, c)$

is barycentric coordinates of incenter I.

c) For $P = H$ we have

$$BA_1 = c\cos B, A_1C = b\cos C, BC_1 = a\cos B, C_1A = b\cos A.$$

Hence, $\dfrac{F_c}{F_b} = \dfrac{BA_1}{A_1C} = \dfrac{c\cos B}{b\cos C} = \dfrac{2R\sin C\cos B}{2R\sin B\cos C} = \dfrac{\tan C}{\tan B}, \dfrac{F_a}{F_b} = \dfrac{BC_1}{C_1A} = \dfrac{a\cos B}{b\cos A} = \dfrac{\tan A}{\tan B} \iff F_a \div F_b \div F_c = \tan A \div \tan B \div \tan C$ and, since $\dfrac{1}{\tan A + \tan B + \tan C}(\tan A, \tan B, \tan C) = \dfrac{1}{\tan A\tan B\tan C}(\tan A, \tan B, \tan C) = (\cot B\cot C, \cot C\cot A, \cot A\cot B)$, then

$$(p_a, p_b, p_c) = (\cot B\cot C, \cot C\cot A, \cot A\cot B)$$

is barycentric coordinates of orthocenter H.

d) For $P = O$ since $\angle BOC = 2A$, $\angle COA = 2B, \angle AOB = 2C$ we have $F_a = \dfrac{R^2\sin 2A}{2}, F_b = \dfrac{R^2\sin 2B}{2}, F_c = \dfrac{R^2\sin 2C}{2}$ and, therefore*, $(p_a, p_b, p_c) = \dfrac{1}{\sin 2A + \sin 2B + \sin 2C}(\sin 2A, \sin 2B, \sin 2C) = \dfrac{1}{4\sin A\sin B\sin C}(\sin 2A, \sin 2B, \sin 2C) =$

$$\left(\frac{\cos A}{\sin B\sin C}, \frac{\cos B}{\sin C\sin A}, \frac{\cos C}{\sin A\sin B}\right)$$

is barycentric coordinates of circumcenter O.

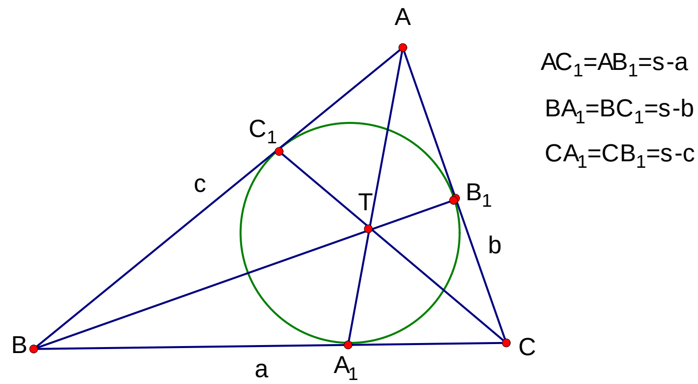
* Note that $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

Problem 2.

a) Let A_1, B_1, C_1 be, respectively, points of tangency of incircle to sides BC, CA, AB of a triangle ABC. Prove that cevians AA_1, BB_1, CC_1 are intersect at one point and find barycentric coordinates of this point.

b) The same questions if A_1, B_1, C_1 be, respectively, points where excircles tangent sides BC, CA, AB.

Solution.



a)

Since $AC_1 = B_1A = s-a,\ C_1B = BA_1 = s-b,\ A_1C = CB_1 = s-c$ then $\dfrac{BA_1}{A_1C}\cdot\dfrac{CB_1}{B_1A}\cdot\dfrac{AC_1}{C_1B} = \dfrac{s-b}{s-c}\cdot\dfrac{s-c}{s-a}\cdot\dfrac{s-a}{s-b} = 1$ and, therefore, by converse of Ceva's Theorem cevians AA_1, BB_1, CC_1 are concurrent. Let T be point of intersection of these cevians. For $P = T$ we have $\dfrac{F_c}{F_b} = \dfrac{BA_1}{A_1C} = \dfrac{s-b}{s-c} = \dfrac{1/(s-c)}{1/(s-b)} = \dfrac{(s-b)(s-a)}{(s-c)(s-a)}, \dfrac{F_a}{F_b} = \dfrac{C_1B}{AC_1} = \dfrac{s-b}{s-a} = \dfrac{1/(s-a)}{1/(s-b)} = \dfrac{(s-b)(s-c)}{(s-c)(s-a)}$.

Hence, $F_a \div F_b \div F_c = (s-b)(s-c) \div (s-c)(s-a) \div (s-a)(s-b) = \dfrac{1}{s-a} \div \dfrac{1}{s-b} \div \dfrac{1}{s-c}$. Let r_a, r_b, r_c be exradii of $\triangle ABC$.Since $r_a(s-a) = r_b(s-b) = r_c(s-c) = F$ and $r_a + r_b + r_c = 4R + r$ then $F_a \div F_b \div F_c = r_a \div r_b \div r_c$ and, therefore,

$$(p_a, p_b, p_c) = \frac{1}{4R+r}(r_a, r_b, r_c)$$

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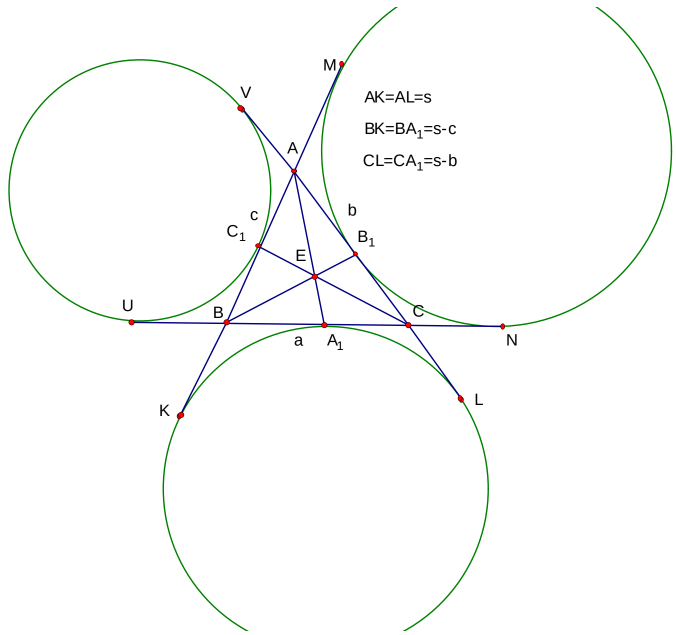


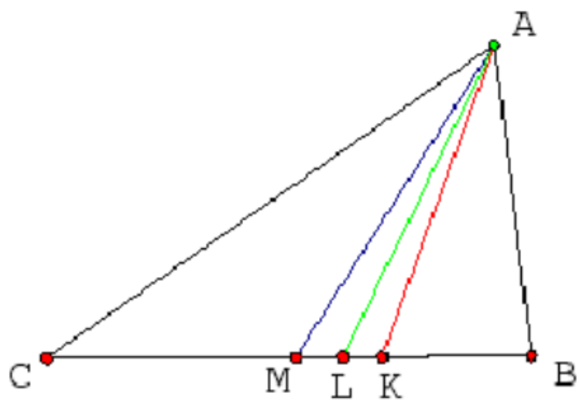
b)

Let $x := BA_1, y := CA_1$.Then $x + y = a,\ AK = AL \iff c + x = b + y$ and, therefore, $2x = x + y + x - y = a + b - c \iff x = s - c,\ y = s - b$ and $AK = AL = s$.Thus $BA_1 = BK = s - c, A_1C = CL = s - b$.Similarly, $B_1A = s - c, AC_1 = s - b$ and $BC_1 = CB_1 = s - a$. Then $\dfrac{BA_1}{A_1C} \cdot \dfrac{CB_1}{B_1A} \cdot \dfrac{AC_1}{C_1B} = \dfrac{s-c}{s-b} \cdot \dfrac{s-a}{s-c} \cdot \dfrac{s-b}{s-a} = 1$ and, therefore, by converse of Ceva's Theorem cevians AA_1, BB_1, CC_1 are concurrent. Let E be point of intersection of these cevians. For $P = E$ we have $\dfrac{F_c}{F_b} = \dfrac{BA_1}{A_1C} = \dfrac{s-c}{s-b}, \dfrac{F_a}{F_b} = \dfrac{C_1B}{AC_1} = \dfrac{s-a}{s-b}$.

Hence, $F_a \div F_b \div F_c = (s-a) \div (s-b) \div (s-c)$ and, therefore, $(p_a, p_b, p_c) = \dfrac{1}{s}(s-a, s-b, s-c)$.

Problem 3.

Find barycentric coordinates of **Lemoine point** (point of intersection of symmedians).($A-$symmedian of triangle ABC is the reflection of the $A-$median in the $A-$internal angle bisector).



pic.1

Let AM, AL, AK be respectively median, angle-bisector and symmedian of $\triangle ABC$ and let $a := BC, b := CA, c := AB, m_a := AM, w_a := AL, k_a := AK, p := ML, q := KL$. Suppose also, that $b \geq c$. Since AL is symmedian in $\triangle ABC$ then AL is angle-bisector in triangle MAK and that imply $\dfrac{m_a}{p} = \dfrac{k_a}{q}$,i.e. there is $t > 0$ such that $k_a = tm_a$ and $q = tp$.Applying Stewart's Formula to chevian AL in triangle MAK we obtain:$w_a^2 = m_a^2 \cdot \dfrac{q}{p+q} + k_a^2 \cdot \dfrac{p}{p+q} - (p+q)^2 \cdot \dfrac{pq}{(p+q)^2} = m_a^2 \cdot \dfrac{q}{p+q} + k_a^2 \cdot \dfrac{p}{p+q} - pq = \dfrac{tm_a^2}{1+t} + \dfrac{k_a^2}{1+t} - tp^2$,because $\dfrac{p}{p+q} = \dfrac{1}{1+t}$, $\dfrac{q}{p+q} = \dfrac{t}{1+t}$.Since AL angle-bisector in $\triangle ABC$ then $CL = \dfrac{ab}{b+c}$ and $p = \dfrac{ab}{b+c} - \dfrac{a}{2} = \dfrac{a(b-c)}{2(b+c)}$. By substitution $w_a^2 = \dfrac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2}$, $m_a^2 = \dfrac{2\left(b^2+c^2\right) - a^2}{4}$, $p = \dfrac{a(b-c)}{2(b+c)}$ and $k_a = tm_a$ in $w_a^2 = \dfrac{tm_a^2}{1+t} + \dfrac{k_a^2}{1+t} - tp^2$ we obtain:

$$\frac{tm_a^2}{1+t} + \frac{k_a^2}{1+t} - tp^2 = \frac{tm_a^2}{1+t} + \frac{t^2m_a^2}{1+t} - tp^2 = t\left(m_a^2 - p^2\right) =$$

$$t\left(\frac{b^2+c^2}{2} - \frac{a^2}{4}\left(1 + \frac{(b-c)^2}{(b+c)^2}\right)\right) = t\left(\frac{b^2+c^2}{2} - \frac{a^2\left(b^2+c^2\right)}{2(b+c)^2}\right) =$$

$$\frac{t\left((b+c)^2 - a^2\right)\left(b^2+c^2\right)}{2(b+c)^2} = \frac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2}.$$

Hence, $t=\dfrac{2bc}{b^2+c^2}$, $k_a=\dfrac{2bcm_a}{b^2+c^2}=\dfrac{bc\sqrt{2\left(b^2+c^2\right)-a^2}}{b^2+c^2}$, $p+q=\dfrac{a\left(b-c\right)}{2\left(b+c\right)}\left(1+t\right)=\dfrac{a\left(b-c\right)}{2\left(b+c\right)}\cdot\dfrac{\left(b+c\right)^2}{b^2+c^2}=\dfrac{a\left(b^2-c^2\right)}{2\left(b^2+c^2\right)}$ and $\dfrac{CK}{KB}=\dfrac{\dfrac{a}{2}+p+q}{\dfrac{a}{2}-\left(p+q\right)}=\dfrac{b^2}{c^2}$.

So, if L is Lemoin's Point (point of intersection of symmedians of $\triangle ABC$) then for barycentric coordinates (L_a, L_b, L_c) of L holds $L_a \div L_b \div L_c = a^2 \div b^2 \div c^2$.

Distances Formulas.

1. Stewart's Formula for length of chevian.

Let $\overrightarrow{OP}=p_a\overrightarrow{OA}+p_b\overrightarrow{OB}$, $p_a+p_b=1$, then $OP^2=\overrightarrow{OP}\cdot\overrightarrow{OP}=$

$$\left(p_a\overrightarrow{OA}+p_b\overrightarrow{OB}\right)\cdot\left(p_a\overrightarrow{OA}+p_b\overrightarrow{OB}\right)=p_a^2OA^2+p_b^2OB^2+2p_ap_b\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)=$$

$$p_a\left(1-p_b\right)OA^2+p_b\left(1-p_a\right)OB^2+2p_ap_b\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)=$$

$$p_aOA^2+p_bOB^2-p_ap_bOA^2-p_ap_bOB^2+2p_ap_b\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)=p_aOA^2+p_bOB^2-p_ap_bAB^2.$$

So, $OP^2=p_aOA^2+p_bOB^2-p_ap_bAB^2$.(Stewart's Formula).

2. Lagrange's Formula.

Let (p_a, p_b, p_c) be baycentric coordinates of the point P, i.e.$p_a+p_b+p_c=1$ and $\overrightarrow{OP}=p_a\overrightarrow{OA}+p_b\overrightarrow{OB}+p_c\overrightarrow{OC}$, then $OP^2=\overrightarrow{OP}\cdot\overrightarrow{OP}=\left(p_a\overrightarrow{OA}+p_b\overrightarrow{OB}+p_c\overrightarrow{OC}\right)\cdot\overrightarrow{OP}=p_a\overrightarrow{OA}\cdot\overrightarrow{OP}+p_b\overrightarrow{OB}\cdot\overrightarrow{OP}+p_c\overrightarrow{OC}\cdot\overrightarrow{OP}=$

$$p_a\overrightarrow{OA}\cdot\left(\overrightarrow{OA}+\overrightarrow{AP}\right)+p_b\overrightarrow{OB}\cdot\left(\overrightarrow{OB}+\overrightarrow{BP}\right)+p_c\overrightarrow{OC}\cdot\left(\overrightarrow{OC}+\overrightarrow{CP}\right)=$$

$$\sum_{cyc}\left(p_aOA^2+p_a\overrightarrow{OA}\cdot\overrightarrow{AP}\right)=\sum_{cyc}p_aOA^2+\sum_{cyc}p_a\left(\overrightarrow{OP}+\overrightarrow{PA}\right)\cdot\overrightarrow{AP}=$$

$$\sum_{cyc}p_aOA^2+\sum_{cyc}p_a\left(\overrightarrow{OP}-\overrightarrow{AP}\right)\cdot\overrightarrow{AP}=\sum_{cyc}p_a\left(OA^2-PA^2\right)+\sum_{cyc}p_a\overrightarrow{OP}\cdot\overrightarrow{AP}=$$

$$\sum_{cyc}p_a\left(OA^2-PA^2\right)+\overrightarrow{OP}\cdot\sum_{cyc}p_a\overrightarrow{AP}=\sum_{cyc}p_a\left(OA^2-PA^2\right)$$

.

So, $OP^2=\sum_{cyc}p_a\left(OA^2-PA^2\right)$ (**Lagrange's formula**).

Remark.

As a corollary from Lagrange's formula we obtain two identities which can be useful.

Let P and be two points on plane with barycentric coordinates (p_a, p_b, p_c) and $Q(q_a, q_b, q_c)$, respectively. Since $QP^2 = \sum_{cyc} p_a \left(QA^2 - PA^2\right)$ and $PQ^2 = \sum_{cyc} q_a \left(PA^2 - QA^2\right)$ we obtain

$$PQ^2 = \frac{1}{2} \sum_{cyc} (p_a - q_a) \left(QA^2 - PA^2\right) \text{ and } \sum_{cyc} (p_a + q_a) \left(PA^2 - QA^2\right) = 0.$$

3. Leibnitz Formula

Let A_1, B_1, C_1 be points intersection of lines PA, PB, PC with BC, CA, AB respectively. Applying Stewart Formula to $O = A_1, P$ and B, C and taking in account that $BA_1 \div CA_1 = p_c \div p_b$ we obtain

$$A_1P^2 = \frac{p_b}{p_b + p_c} PB^2 + \frac{p_c}{p_b + p_c} PC^2 - \frac{p_b}{p_b + p_c} \cdot \frac{p_c}{p_b + p_c} a^2$$

and, and since $\overrightarrow{A_1P} = -\frac{p_a}{p_b + p_c} \overrightarrow{AP}$ then $A_1P^2 = \frac{p_a^2}{(p_b + p_c)^2} AP^2$.

Therefore, $\frac{p_a^2}{(p_b + p_c)^2} AP^2 = \frac{p_b}{p_b + p_c} PB^2 + \frac{p_c}{p_b + p_c} PC^2 - \frac{p_b}{p_b + p_c} \cdot \frac{p_c}{p_b + p_c} a^2 \iff$ $p_a^2 AP^2 = p_b (p_b + p_c) PB^2 + p_c (p_b + p_c) PC^2 - p_b p_c a^2$.Hence, $\sum_{cyc} p_a^2 AP^2 = \sum_{cyc} p_b (p_b + p_c) PB^2 + \sum_{cyc} p_c (p_b + p_c) PC^2 - \sum_{cyc} p_b p_c a^2 \iff$

$$\sum_{cyc} p_b p_c a^2 = \sum_{cyc} \left(p_b^2 + p_b p_c\right) PB^2 + \sum_{cyc} \left(p_b p_c + p_c^2\right) PC^2 - \sum_{cyc} p_a^2 AP^2 =$$

$$\sum_{cyc} p_b^2 PB^2 + \sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_b p_c PC^2 + \sum_{cyc} p_c^2 PC^2 - \sum_{cyc} p_a^2 AP^2 =$$

$$\sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_b p_c PC^2 + \sum_{cyc} p_c^2 PC^2 = \sum_{cyc} p_b p_c PB^2 + \sum_{cyc} p_c p_a PA^2 + \sum_{cyc} p_c^2 PC^2 =$$

$$\sum_{cyc} p_c \left(p_b PB^2 + p_a PA^2 + p_c PC^2\right) = \left(p_b PB^2 + p_a PA^2 + p_c PC^2\right) \sum_{cyc} p_c = \sum_{cyc} p_a PA^2$$

Thus, $\sum_{cyc} p_a PA^2 = \sum_{cyc} p_b p_c a^2$ and, therefore, $OP^2 = \sum_{cyc} p_a \left(OA^2 - PA^2\right) \iff$

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 \text{ } \textbf{(Leibnitz Formula).}$$

Application of distance formulas.

1. Distance between circumcenter O and centroid G.

Let O be circumcenter, $R-$circumradius and $P = G\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$,then $OG^2 = \sum\limits_{cyclic} \dfrac{1}{3} \cdot \left(R^2 - GA^2\right) = R^2 - \dfrac{1}{3} \sum\limits_{cyclic} GA^2$.

Since $GA^2 = \dfrac{4}{9}\left(\dfrac{2\left(b^2+c^2\right)-a^2}{4}\right) = \dfrac{2\left(b^2+c^2\right)-a^2}{9}$ then $\sum\limits_{cyclic} GA^2 = \dfrac{a^2+b^2+c^2}{3}$ and $OG^2 = R^2 - \dfrac{a^2+b^2+c^2}{9}$.

This imply $R^2 - \dfrac{a^2+b^2+c^2}{9} \geq 0 \iff a^2+b^2+c^2 \leq 9R^2$.

2. Distance between circumcenter O and incenter I.
(Euler's formula and Euler's inequality).

Let O be circumcenter. Since $I\left(\dfrac{a}{a+b+c}, \dfrac{b}{a+b+c}, \dfrac{c}{a+b+c}\right)$, then $(a+b+c)\,OI^2 = \sum\limits_{cyc} a\left(OA^2 - IA^2\right) = \sum\limits_{cyc} a\left(R^2 - IA^2\right) = (a+b+c)\,R^2 - \sum\limits_{cyc} aIA^2$.

Since $aIA^2 = \dfrac{aw_a^2\,(b+c)^2}{(a+b+c)^2} = \dfrac{abc\,(a+b+c)\,(b+c-a)\,(b+c)^2}{(a+b+c)^2\,(b+c)^2} = \dfrac{abc\,(b+c-a)}{a+b+c}$ then $\sum\limits_{cyclic} aIA^2 = abc$ and $OI^2 = R^2 - \dfrac{abc}{a+b+c} = R^2 - \dfrac{4Rrs}{2s} = R^2 - 2Rr$.

Hence, $OI = \sqrt{R^2 - 2Rr}$ and $R^2 - 2Rr \geq 0 \iff R \geq 2r$.

Remark.

Consider now general situation, when O be circumcenter, $R-$circumradius of circumcircle of $\triangle ABC$ and (p_a, p_b, p_c) is barycentric coordinates of some point P.Then applying general Leibnitz Formula for such origin O we obtain:

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 = \sum_{cyc} p_a R^2 - \sum_{cyc} p_b p_c a^2 = R^2 - \sum_{cyc} p_b p_c a^2.$$

Thus $\sum\limits_{cyc} p_b p_c a^2 \leq R^2$ and $OP = \sqrt{R^2 - \sum\limits_{cyc} p_b p_c a^2}$.

Using the formula obtained for the OP, we consider several more cases of calculating the distances between circumcenter O and another triangle centers..

But for beginning we will apply this formula for considered above two cases.

If $P = G\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$ then $\sum\limits_{cyc} p_b p_c a^2 = \dfrac{1}{9}\sum\limits_{cyc} a^2$ and, therefore,

$$OG = \sqrt{R^2 - \frac{a^2+b^2+c^2}{9}}$$

;

If $P = I\left(\frac{a}{2s},\frac{b}{2s},\frac{c}{2s}\right)$ then $\sum\limits_{cyc} p_b p_c a^2 = \frac{1}{4s^2}\sum\limits_{cyc} bca^2 = \frac{abc\,(a+b+c)}{4s^2} = \frac{4Rrs\cdot 2s}{4s^2} = 2Rr$ and, therefore,

$$OI = \sqrt{R^2 - 2Rr}$$

.

3. Distance between circumcenter O and orthocenter H.

Since $H\,(\cot B\cot C, \cot C\cot A, \cot A\cot B)$ then $\sum\limits_{cyc} p_b p_c a^2 = \sum\limits_{cyc}\cot C\cot A\cdot\cot A\cot B\cdot a^2 = \cot A\cot B\cot C\sum\limits_{cyc} a^2\cot A$. Noting that $\sum\limits_{cyc}\cot A\cdot a^2 = 4R^2\sum\limits_{cyc}\cot A\cdot\sin^2 A = 2R^2\sum\limits_{cyc}\sin 2A = 8R^2\sin A\sin B\sin C$ and $\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$ we obtain $\sum\limits_{cyc} p_b p_c a^2 = \cot A\cot B\cot C\sum\limits_{cyc} a^2\cot A = \cot A\cot B\cot C\cdot 8R^2\sin A\sin B\sin C = 8R^2\cos A\cos B\cos C = 8R^2\cdot\frac{s^2-(2R+r)^2}{4R^2} = 2\left(s^2-(2R+r)^2\right)$ and, therefore,

$$OH = \sqrt{R^2 - 2\left(s^2-(2R+r)^2\right)} = \sqrt{9R^2+8Rr+2r^2-2s^2}.$$

And by the way we obtain inequality $s^2 \le \frac{9R^2+8Rr+2r^2}{2}$.

Remark.

This inequality also immediately follows from Gerretsen's Inequality $s^2 \le 4R^2+4Rr+3r^2$ and Euler's Inequality $R \ge 2r$. Indeed, $9R^2+8Rr+2r^2-2s^2 \ge 9R^2+8Rr+2r^2-2\left(4R^2+4Rr+3r^2\right) = (R-2r)(R+2r)$).

4. Distance between circumcenter O and point T.(see Problem 2a. in Application1)

Since for $P = T$ we have $(p_a,p_b,p_c) = \left(\frac{1}{k(s-a)},\frac{1}{k(s-b)},\frac{1}{k(s-c)}\right)$,

where $k = \sum\limits_{cyc}\frac{1}{s-a} = \frac{4R+r}{sr}$ then[*] $\sum\limits_{cyc} p_b p_c a^2 = \frac{1}{k^2}\sum\limits_{cyc}\frac{a^2}{(s-b)(s-c)} = \frac{s^2r^2}{(4R+r)^2(s-a)(s-b)(s-c)}\sum\limits_{cyc} a^2(s-a) = \frac{s^2r^2}{(4R+r)^2 sr^2}\sum\limits_{cyc} a^2(s-a) = \frac{s}{(4R+r)^2}\sum\limits_{cyc} a^2(s-a) = \frac{4s^2r\,(R+r)}{(4R+r)^2}$ and,therefore,

$$OT = \sqrt{R^2 - \frac{4s^2r\,(R+r)}{(4R+r)^2}}.$$

And by the way we obtain inequality $s^2 \le \frac{R^2(4R+r)^2}{4r\,((R+r))}$,which also can be proved using Gerretsen's Inequality $s^2 \leq 4R^2+4Rr+3r^2$ and Euler's Inequality $R \geq 2r$.

* Since $ab+bc+ca = s^2+4Rr+r^2$, $a^2+b^2+c^2 = 4s^2-2(ab+bc+ca) = 2(s^2-4Rr-r^2)$, $a^3+b^3+c^3 = 3abc+(a+b+c)^3-3(a+b+c)(ab+bc+ca) = 3\cdot 4Rrs+8s^3-6s(s^2+4Rr+r^2) = 2s(s^2-6Rr-3r^2)$ we obtain

$$\sum_{cyc} a^2(s-a) = 2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2) = 4rs(R+r)$$

5. Distance between circumcenter O and point E (see Problem 2b. in Application1)

Since for $P=E$ we have $(p_a,p_b,p_c) = \dfrac{1}{s}(s-a,s-b,s-c)$ then $\sum_{cyc} p_bp_ca^2 = \dfrac{1}{s^2}\sum_{cyc}(s-b)(s-c)a^2 = \dfrac{1}{s^2}\sum_{cyc}(a^2s^2-a^2s(b+c)+a^2bc) = a^2+b^2+c^2+\dfrac{abc(a+b+c)}{s^2}-\dfrac{(a+b+c)(ab+bc+ca)}{s}+\dfrac{3abc}{s} = 2(s^2-4Rr-r^2)+8Rr-2(s^2+4Rr+r^2)+12Rr = 4r(R-r)$ and,therefore, $OE = \sqrt{R^2-4r(R-r)} = R-2r$ and, by the way, our calculation of QE give us one more proof of Euler's Inequality.

6. Distance between circumcenter O and point L (Lemioin′s point).

Since for $P=L$ we have $(p_a,p_b,p_c) = \dfrac{1}{a^2+b^2+c^2}(a^2,b^2,c^2)$ then $\sum_{cyc} p_bp_ca^2 = \dfrac{1}{(a^2+b^2+c^2)^2}\sum_{cyc} b^2c^2\cdot a^2 = \dfrac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}$ and, therefore, $OL = \sqrt{R^2-\dfrac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}} = \sqrt{R^2-\dfrac{48R^2r^2s^2}{(a^2+b^2+c^2)^2}} = R\sqrt{1-\dfrac{48F^2}{(a^2+b^2+c^2)^2}}$ and, by the way, our calculation of QL give us one more proof of Weitzenböck's inequality $a^2+b^2+c^2 \geq 4\sqrt{3}F$.

Remark.

Since $(a^2+b^2+c^2)^2-48F^2 = (a^2+b^2+c^2)^2-3(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4) = 4(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)$ then

$$OL = 2R\sqrt{\frac{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}{(a^2+b^2+c^2)^2}}$$

Problem 4.

Let ABC be a triangle with sidelengths a,b,c and let M be any point lying on circumcircle of $\triangle ABC$.Find the maximum and minimum of the the following expression:

a) $a\cdot MA^2+b\cdot MB^2+c\cdot MC^2$(All Israel Math Olympiad);.

★b) $\tan A\cdot MA^2+\tan B\cdot MB^2+\tan C\cdot MC^2$ if $\triangle ABC$ is acute angled triangle;

★**c)** $\sin 2A\cdot MA^2+\sin 2B\cdot MB^2+\sin 2C\cdot MC^2$;
★**d)** $a^2\cdot MA^2+b^2\cdot MB^2+c^2\cdot MC^2$;
★**e)** $\dfrac{MA^2}{s-a}+\dfrac{MB^2}{s-b}+\dfrac{MC^2}{s-c}$.
★**f)** $(s-a)\,MA^2+(s-b)\,MB^2+(s-c)\,MC^2$

Solution.

First we consider a common approach to the all these problems represented in the following general formulation:

Let α,β,γ be real numbers such that $\alpha+\beta+\gamma\neq 0$ and let M be any point lying on circumcircle of a triangle ABC with sidelengths a,b,c and circumradius R

Find the maximal and the minimal values of the expression:

$$D(M):=\ \alpha\cdot MA^2+\beta\cdot MB^2+\gamma\cdot MC^2.$$

Let P be a point on the plane with barycentric coordinates $(p_a,p_b,p_c)=\dfrac{1}{\alpha+\beta+\gamma}(\alpha,\beta,\gamma)$. Then, by replacing origin O in the Leibnitz Formula with M, we obtain

$$MP^2=\sum_{cyc}p_aMA^2-\sum_{cyc}p_bp_ca^2=\frac{1}{\alpha+\beta+\gamma}\sum_{cyc}\alpha\cdot MA^2-\frac{1}{(\alpha+\beta+\gamma)^2}\sum_{cyc}\beta\gamma a^2\iff$$

$$D(M)=(\alpha+\beta+\gamma)\,MP^2+\frac{1}{\alpha+\beta+\gamma}\sum_{cyc}\beta\gamma a^2=(\alpha+\beta+\gamma)\left(MP^2+\sum_{cyc}p_bp_ca^2\right).$$

Since $\sum_{cyc}p_bp_ca^2$ isn't depend from M then the problem reduces to finding the largest and smallest value of $(\alpha+\beta+\gamma)\,MP^2$. Wherein if $\alpha+\beta+\gamma<0$ then $\max\left((\alpha+\beta+\gamma)\,MP^2\right)=(\alpha+\beta+\gamma)\min MP^2$ and
$\min\left((\alpha+\beta+\gamma)\,MP^2\right)=(\alpha+\beta+\gamma)\max MP^2$.

Bearing in mind the application of the general case to the problems listed above, and also not to overload the text, we assume further that $\alpha+\beta+\gamma>0$ and that point P is interior with respect to circumcircle. Hence,

Then if d is the distant between point P and circumcenter O then $\max MP=R+d$ and $\min MP=R-d$.

$$\max D(M)=(\alpha+\beta+\gamma)\left((R+d)^2+\sum_{cyc}p_bp_ca^2\right)$$

and

$$\min D(M)=(\alpha+\beta+\gamma)\left((R-d)^2+\sum_{cyc}p_bp_ca^2\right).$$

.

Coming back to the listed above subproblems we obtain:

a) Since $(\alpha,\beta,\gamma\ )=(a,b,c)\,,P=I,(p_a,p_b,p_c)=\left(\dfrac{a}{2s},\dfrac{b}{2s},\dfrac{c}{2c}\right),\ d=OI=\sqrt{R^2-2Rr}$ and $\sum\limits_{cyc}p_bp_ca^2\ =2Rr$ (see **Distance between circumcenter** O **and incenter** I) then for $D(M)=a\cdot MA^2+b\cdot MB^2+c\cdot MC^2$ we obtain
$\max D(M)=(a+b+c)\left(\left(R+\sqrt{R^2-2Rr}\right)^2+2Rr\right)=4Rs\left(R+\sqrt{R^2-2Rr}\right)$ and
$\min D(M)=(a+b+c)\left(\left(R-\sqrt{R^2-2Rr}\right)^2+2Rr\right)=4Rs\left(R-\sqrt{R^2-2Rr}\right).$

b) Since

$(\alpha,\beta,\gamma\ )=(\tan A,\tan B,\tan C)\,,\ (p_a,p_b,p_c)=(\cot B\cot C,\cot C\cot A,\cot A\cot B)\,,$

$d=OH=\sqrt{9R^2+8Rr+2r^2-2s^2},\ \tan A+\tan B+\tan C=\dfrac{2sr}{s^2-(2R+r)^2}$

and $\sum\limits_{cyc}p_bp_ca^2\ =2\left(s^2-(2R+r)^2\right)$ (see **Distance between circumcenter** O **and orthocenter** H) then for

$$D(M)=\tan A\cdot MA^2+\tan B\cdot MB^2+\tan C\cdot MC^2$$

we we obtain

$$\max D(M)=(\tan A+\tan B+\tan C)\left(\left(R+\sqrt{9R^2+8Rr+2r^2-2s^2}\right)^2+2\left(s^2-(2R+r)^2\right)\right)=$$

$$\frac{2sr}{s^2-(2R+r)^2}\cdot 2R\left(R+\sqrt{9R^2+8Rr+2r^2-2s^2}\right)=\frac{4Rrs\left(R+\sqrt{9R^2+8Rr+2r^2-2s^2}\right)}{s^2-(2R+r)^2}$$

and

$$\min D(M)=\frac{4Rrs\left(R-\sqrt{9R^2+8Rr+2r^2-2s^2}\right)}{s^2-(2R+r)^2}$$

.

c) Since

$$(\alpha,\beta,\gamma\ )=(\sin 2A,\sin 2B,\sin 2C)\,,\ P=O,$$
$$(p_a,p_b,p_c)=\left(\frac{\cos A}{\sin B\sin C},\frac{\cos B}{\sin C\sin A},\frac{\cos C}{\sin A\sin B}\right)$$

and and $d=OO=0$ then

$$D(M)=\sin 2A\cdot MA^2+\sin 2B\cdot MB^2+\sin 2C\cdot MC^2=(\sin 2A+\sin 2B+\sin 2C)\sum_{cyc}\frac{\cos B}{\sin C\sin A}\cdot\frac{\cos C}{\sin A\sin B}a^2=$$

$$4\sin A\sin B\sin C\sum_{cyc}\frac{a^2\cos B\cos C}{\sin^2 A\sin C\sin B}=4\sum_{cyc}\frac{a^2\cos B\cos C}{\sin A}=8R^2\sum_{cyc}\sin A\cos B\cos C.$$

That is for any point M that lies on circumcircle $D(M)$ is the constant, namely

$$\sum_{cyc} \sin 2A \cdot MA^2 = 8R^2 \sum_{cyc} \sin A \cos B \cos C.$$

d) Since

$$(\alpha, \beta, \gamma) = \left(a^2, b^2, c^2\right), \ P = L, \ (p_a, p_b, p_c) = \frac{1}{a^2+b^2+c^2}\left(a^2, b^2, c^2\right),$$

$$d = OL = R\sqrt{1 - \frac{48F^2}{\left(a^2+b^2+c^2\right)^2}}, \ \sum_{cyc} p_b p_c a^2 = \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2} = \frac{48R^2F^2}{\left(a^2+b^2+c^2\right)^2}$$

(see **Distance between circumcenter** O **and Lemoin point** L) then for

$$D(M) = a^2 \cdot MA^2 + b^2 \cdot MB^2 + c^2 \cdot MC^2$$

we obtain

$$\max D(M) = \left(a^2+b^2+c^2\right)\left(R^2\left(1+\sqrt{1-\frac{48F^2}{\left(a^2+b^2+c^2\right)^2}}\right)^2 + \frac{48R^2F^2}{\left(a^2+b^2+c^2\right)^2}\right) =$$

$$\frac{R^2}{a^2+b^2+c^2}\left(\left(a^2+b^2+c^2+\sqrt{\left(a^2+b^2+c^2\right)^2-48F^2}\right)^2+48F^2\right) =$$

$$2R^2\left(2\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}+a^2+b^2+c^2\right)$$

because $\left(a^2+b^2+c^2\right)^2 - 48F^2 = 4\left(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2\right)$ and $\left(t+\sqrt{t^2-48F^2}\right)^2 + 48F^2 = 2t\left(\sqrt{t^2-48F^2}+t\right)$,where $t = a^2+b^2+c^2$.

Also,

$$\min D(M) = \left(a^2+b^2+c^2\right)\left(R^2\left(1-\sqrt{1-\frac{48F^2}{\left(a^2+b^2+c^2\right)^2}}\right)^2 + \frac{48R^2F^2}{\left(a^2+b^2+c^2\right)^2}\right) =$$

$$\frac{R^2}{a^2+b^2+c^2}\left(\left(a^2+b^2+c^2-\sqrt{\left(a^2+b^2+c^2\right)^2-48F^2}\right)^2+48F^2\right) =$$

$$2R^2\left(a^2+b^2+c^2-2\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}\right)$$

e) Since $(\alpha, \beta, \gamma) = \left(\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}\right)$, $P = T, (p_a, p_b, p_c) = \left(\frac{1}{k(s-a)}, \frac{1}{k(s-b)}, \frac{1}{k(s-c)}\right)$,

where $k=\sum_{cyc}\frac{1}{s-a}=\frac{4R+r}{sr},\ d=OT=\sqrt{R^2-\frac{4s^2r(R+r)}{(4R+r)^2}}$ and
$\sum_{cyc}p_bp_ca^2=\frac{4s^2r(R+r)}{(4R+r)^2}$ (see **Distance between circumcenter O and T**) then for $D(M)=\frac{MA^2}{s-a}+\frac{MB^2}{s-b}+\frac{MC^2}{s-c}$ we obtain

$$\max D(M)=\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\left(\left(R+\sqrt{R^2-\frac{4s^2r(R+r)}{(4R+r)^2}}\right)^2+\frac{4s^2r(R+r)}{(4R+r)^2}\right)=$$

$$\frac{4R+r}{sr}\cdot 2R\left(R+\frac{\sqrt{R^2(4R+r)^2-4rs^2(R+r)}}{4R+r}\right)=\frac{2R\left(R(4R+r)+\sqrt{R^2(4R+r)^2-4rs^2(R+r)}\right)}{sr}$$

and

$$\min D(M)=\frac{2R\left(R(4R+r)-\sqrt{R^2(4R+r)^2-4rs^2(R+r)}\right)}{sr}$$

f) Since $(\alpha,\beta,\gamma)=(s-a,s-b,s-c),\ P=E,(p_a,p_b,p_c)=\frac{1}{s}(s-a,s-b,s-c)$,
$\sum_{cyc}p_bp_ca^2=4r(R-r),d=OE=R-2r$ (see **Distance between circumcenter O and E**) then for $D(M)=(s-a)MA^2+(s-b)MB^2+(s-c)MC^2$ we obtain

$$\max D(M)=s\left((R+R-2r)^2+4r(R-r)\right)=4sR(R-r)$$

and

$$\min D(M)=s\left((R-(R-2r))^2+4r(R-r)\right)=4Rsr=abc.$$

Problem 5.

Let a,b,c be sidelengths of a triangle ABC. Find point O in the plane such that the sum

$$\frac{OA^2}{b^2}+\frac{OB^2}{c^2}+\frac{OC^2}{a^2}$$

is minimal.

Solution.

Let P be point on the plane with barycentric coordinates $(p_a, p_b, p_c) = \left(\frac{1}{kb^2}, \frac{1}{kc^2}, \frac{1}{ka^2}\right)$,where $k = \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a^2}$.

Then by Leibniz Formula

$$OP^2 = \sum_{cyc} p_a OA^2 - \sum_{cyc} p_b p_c a^2 = \frac{1}{k}\sum_{cyc}\frac{OA^2}{b^2} - \frac{1}{k^2}\sum_{cyc}\frac{1}{c^2a^2}\cdot a^2 =$$

$$\frac{1}{k}\sum_{cyc}\frac{OA^2}{b^2} - \frac{1}{k^2}\sum_{cyc}\frac{1}{c^2} = \frac{1}{k}\left(\sum_{cyc}\frac{OA^2}{b^2} - 1\right).$$

Hence, $\sum_{cyc}\frac{OA^2}{b^2} = k \cdot OP^2 + 1$ and, therefore, $\min \sum_{cyc}\frac{OA^2}{b^2} = 1 = \sum_{cyc}\frac{PA^2}{b^2}$. That is $\sum_{cyc}\frac{OA^2}{b^2}$ is minimal iff $O = P$, where P is intersect point of cevians AA_1, BB_1, CC_1 such that $\frac{BA_1}{A_1C} = \frac{F_c}{F_b} = \frac{p_c}{p_b} = \frac{c^2}{a^2}, \frac{CB_1}{B_1A} = \frac{p_a}{p_c} = \frac{a^2}{b^2}, \frac{AC_1}{C_1B} = \frac{p_b}{p_a} = \frac{b^2}{c^2}$.

Problem 6. Let ABC be a triangle with sidelengths $a = BC, b = CA, c = AB$ and let s, R and r be semiperimeter, circumradius and inradius of $\triangle ABC$, respectively.

For any point P lying on incircle of $\triangle ABC$ let

$$D(P) := aPA^2 + bPB^2 + cPC^2.$$

Prove that $D(P)$ is a constant and find its value in terms of s, R and r.

Solution.

Let I be incener of $\triangle ABC$ and let (i_a, i_b, i_c) be baricentric coordinates of I. Since $(i_a, i_b, i_c) = \frac{1}{2s}(a, b, c)$ and $PI = r$ then applying Leibniz Formula for distance between points I and P we obtain $r^2 = PI^2 = \sum_{cyc} i_a \cdot PA^2 - \sum_{cyc} i_b i_c a^2 = \frac{1}{2s}\sum aPA^2 - \frac{1}{4s^2}\sum_{cyc} bca^2 = \frac{1}{2s}\sum_{cyc} aPA^2 - \frac{abc\cdot 2s}{4s^2} = \frac{1}{2s}\sum_{cyc} aPA^2 - \frac{4Rrs}{2s} = \frac{1}{2s}\sum aPA^2 - 2Rr$.

Hence, $\sum_{cyc} aPA^2 = 2s\left(r^2 + 2Rr\right)$.

Area of a triangle, equation of a line and equation of a circle in barycentric coordinates.

1. **Area of a triangle.**

First we recall that for any two vectors $\mathbf{a}, \mathbf{b}$ on the plane is defined skew product $\mathbf{a} \wedge \mathbf{b} := \|\mathbf{a}\| \|\mathbf{b}\| \sin\left(\widehat{\mathbf{a}, \mathbf{b}}\right)$ and if (a_1, a_2) , (b_1, b_2) are Cartesian coordinates of $\mathbf{a}, \mathbf{b}$,respectively, then

$$\mathbf{a} \wedge \mathbf{b} = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1.$$

Geometrically $\mathbf{a}\wedge\mathbf{b}$ is oriented (because $\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}$) area of parallelogram defined by vectors $\mathbf{a},\mathbf{b}$.Obvious that $\mathbf{a}\wedge\mathbf{b}=0$ iff $\mathbf{a},\mathbf{b}$ are collinear (in particular $\mathbf{a}\wedge\mathbf{a}=0$ for any $\mathbf{a}$).

Using coordinate definition of skew product easy to prove that it is bilinear, that is $(\mathbf{a}+\mathbf{b})\wedge\mathbf{c}=\mathbf{a}\wedge\mathbf{c}+\mathbf{b}\wedge\mathbf{c}$ (then also $\mathbf{a}\wedge(\mathbf{c}+\mathbf{b})=-(\mathbf{c}+\mathbf{b})\wedge\mathbf{a}=-(\mathbf{c}\wedge\mathbf{a}+\mathbf{b}\wedge\mathbf{a})=(-\mathbf{c}\wedge\mathbf{a})+(-\mathbf{b}\wedge\mathbf{a})=\mathbf{a}\wedge\mathbf{c}+\mathbf{a}\wedge\mathbf{b}$) and$(p\mathbf{a})\wedge\mathbf{b}=\mathbf{a}\wedge(p\mathbf{b})=p(\mathbf{a}\wedge\mathbf{b})$ for any real p.

For any three point K,L,M on the plane which are not collinear we will use common notation $[K,L,M]$ for oriented area of $\triangle KLM$ which equal to $\frac{1}{2}\overrightarrow{KL}\wedge\overrightarrow{KM}$ (in the case if K,L,M are collinear we obtain $[K,L,M]=0$). Regular area of $\triangle KLM$ is $\frac{1}{2}\left|\overrightarrow{KL}\wedge\overrightarrow{KM}\right|$.

Let P,Q,R be three point on the plane and $(p_a,p_b,p_c)\,,(q_a,q_b,q_c)\,,(r_a,r_b,r_c)$ be, respectively their barycentric coordinates with respect to triangle ABC.Then $\overrightarrow{AP}=p_a\overrightarrow{AA}+p_b\overrightarrow{AB}+p_c$ and, similarly, $\overrightarrow{AQ}=q_b\overrightarrow{AB}+q_c\overrightarrow{AC}$, $\overrightarrow{AR}=r_b\overrightarrow{AB}+r_c\overrightarrow{AC}$.

Hence, $\overrightarrow{PQ}=(q_b-p_b)\overrightarrow{AB}+(q_c-p_c)\overrightarrow{AC}$, $\overrightarrow{PR}=(r_b-p_b)\overrightarrow{AB}+(r_c-p_c)\overrightarrow{AC}$ and, therefore,

$$2[P,Q,R]=\overrightarrow{PQ}\wedge\overrightarrow{PR}=\left((q_b-p_b)\overrightarrow{AB}+(q_c-p_c)\overrightarrow{AC}\right)\wedge\left((r_b-p_b)\overrightarrow{AB}+(r_c-p_c)\overrightarrow{AC}\right)=$$

$$(q_b-p_b)(r_c-p_c)\overrightarrow{AB}\wedge\overrightarrow{AC}+(q_c-p_c)(r_b-p_b)\overrightarrow{AC}\wedge\overrightarrow{AB}=$$

$$((q_b-p_b)(r_c-p_c)-(r_b-p_b)(q_c-p_c))\overrightarrow{AB}\wedge\overrightarrow{AC}=2[A,B,C]\cdot\det\begin{pmatrix}q_b-p_b & r_b-p_b\\ q_c-p_c & r_c-p_c\end{pmatrix}.$$

Thus,

$$[P,Q,R]=\det\begin{pmatrix}q_b-p_b & r_b-p_b\\ q_c-p_c & r_c-p_c\end{pmatrix}\cdot[A,B,C].$$

Or, since

$$\det\begin{pmatrix}q_b-p_b & r_b-p_b\\ q_c-p_c & r_c-p_c\end{pmatrix}=(q_b-p_b)(r_c-p_c)-(r_b-p_b)(q_c-p_c)=$$

$$p_bq_c+p_cr_b+q_br_c-p_cq_b-p_br_c-q_cr_b=\det\begin{pmatrix}1 & p_b & p_c\\ 1 & q_b & q_c\\ 1 & r_b & r_c\end{pmatrix}=\det\begin{pmatrix}p_a & p_b & p_c\\ q_a & q_b & q_c\\ r_a & r_b & r_c\end{pmatrix}$$

(because $1-p_b-p_c=p_a,1-q_b-q_c=q_a,1-r_b-r_c=r_a$) and, therefore, we obtain more representative form of obtained correlation (Areas Formula)

$$\textbf{(AF)}\qquad [P,Q,R]=\det\begin{pmatrix}p_a & p_b & p_c\\ q_a & q_b & q_c\\ r_a & r_b & r_c\end{pmatrix}[A,B,C].$$

Using this formula we can to do important conclusion, namely:

Points P, Q, R are collinear iff $\det\begin{pmatrix} p_a & p_b & p_c \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0.$

From that immediately follows that set of points on the plane with barycentric coordinates (x, y, z) such that $\det\begin{pmatrix} x & y & z \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0$ is line which passed through points $Q(q_a, q_b, q_c)$ and $R(r_a, r_b, r_c)$,that is $\det\begin{pmatrix} x & y & z \\ q_a & q_b & q_c \\ r_a & r_b & r_c \end{pmatrix} = 0$ is equation of line in baycentric coordinates.

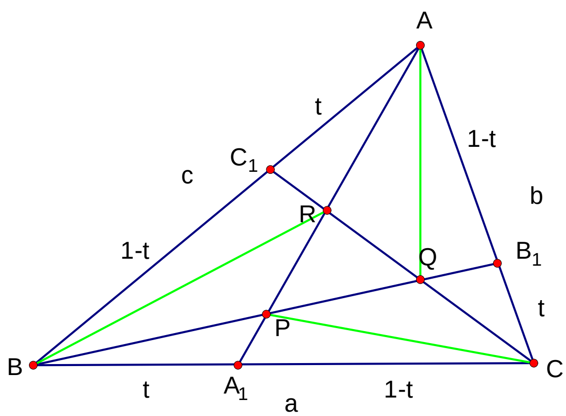
As another application of formula (AF) we will solve the following

Problem 7:

Let AA_1, BB_1, CC_1 be cevians of a triangle ABC such that $\dfrac{AB_1}{B_1C} = \dfrac{CA_1}{A_1B} = \dfrac{BC_1}{C_1A} = \dfrac{1-t}{t}$.

Find the ratio $\dfrac{[P, Q, R]}{[A, B, C]}$.

Solution.



Let $(p_a, p_b, p_c), (q_a, q_b, q_c), (r_a, r_b, r_c)$ be, respectively, barycentric coordinates of points P, Q, R .Then $\dfrac{A_1B}{A_1C} = \dfrac{t}{1-t} = \dfrac{p_c}{p_b}, \dfrac{B_1C}{B_1A} = \dfrac{t}{1-t} = \dfrac{p_a}{p_c}$.

Noting that $\dfrac{p_a}{p_c} = \dfrac{t}{1-t} = \dfrac{t^2}{t(1-t)}, \dfrac{p_b}{p_c} = \dfrac{1-t}{t} = \dfrac{(1-t)^2}{t(1-t)}$ we can conclude that $p_a = kt^2, p_b = k(1-t)^2$, $p_c = kt(1-t)$, for some k and since $p_a + p_b + p_c = 1$ we obtain $k\left(t^2 + (1-t)^2 + t(1-t)\right) = 1 \iff k\left(t^2 - t + 1\right) = 1 \iff k = \dfrac{1}{t^2 - t + 1}$.

Hence,

$$p_a = \frac{t^2}{t^2 - t + 1}, p_b = \frac{(1-t)^2}{t^2 - t + 1}, p_c = \frac{t(1-t)}{t^2 - t + 1}.$$

Since $\dfrac{q_c}{q_a} = \dfrac{1-t}{t}$ and $\dfrac{q_b}{q_a} = \dfrac{t}{1-t}$ we, as above, obtain

$$q_a = \frac{t(1-t)}{t^2 - t + 1} = p_c, q_b = \frac{t^2}{t^2 - t + 1} = p_a, q_c = \frac{(1-t)^2}{t^2 - t + 1} = p_b,$$

that is $(q_a, q_b, q_c) = (p_c, p_a, p_b)$ and, similarly, $(r_a, r_b, r_c) = (p_b, p_c, p_a)$.
Hence,

$$\frac{[P, Q, R]}{[A, B, C]} = \det \begin{pmatrix} p_a & p_b & p_c \\ p_c & p_a & p_b \\ p_b & p_c & p_a \end{pmatrix} =$$

$$p_a^3 + p_b^3 + p_c^3 - 3p_a p_b p_c = (p_a + p_b + p_c)^3 - 3(p_a + p_b + p_c)(p_a p_b + p_b p_c + p_c p_a) =$$

$$1 - 3(p_a p_b + p_b p_c + p_c p_a) = \frac{1}{(t^2 - t + 1)^2}\left(t^2(1-t)^2 + (1-t)^3 t + t^3(1-t)\right) =$$

$$\frac{t(1-t)\left(t(1-t) + (1-t)^2 + t^2\right)}{(t^2 - t + 1)^2} = \frac{t(1-t)}{t^2 - t + 1}.$$

Equation of a circle in barycentric coordinates.

Let O be center of a circle with radius R. And let P be any point on lying on this circle. If (o_a, o_b, o_c) and $(p_a, p_b, p_c) = (x, y, z)$ be, respectively, barycentric coordinates of O and P then

by Leybnitz Formula $OP^2 = \sum\limits_{cyc} p_a OA^2 - \sum\limits_{cyc} p_b p_c a^2 \iff$

(EC) $\qquad R^2 = xOA^2 + yOB^2 + zOC^2 - yza^2 - zxb^2 - xyc^2.$

In particular, if O and R be circumcenter and circumradius of $\triangle ABC$ then $xOA^2 + yOB^2 + zOC^2 = R^2(x + y + z) = R^2$ and, therefore,

(**ECc**) $\qquad yza^2 + zxb^2 + xyc^2 = 0$

is equation of circumcircle of $\triangle ABC$.

By replacing O and R in **(EC)** with I (incenter) and r (inradius) we obtain $r^2 = xIA^2 + yIB^2 + zIC^2 - yza^2 - zxb^2 - xyc^2$.Since $IA = \dfrac{b+c}{a+b+c} \cdot l_a$, where l_a is length of angle bisector from A and $l_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$ then $IA^2 = \dfrac{(b+c)^2}{4s^2} \cdot \dfrac{4bcs(s-a)}{(b+c)^2} = \dfrac{bc(s-a)}{s}$ and, cyclic, $IB^2 = \dfrac{ca(s-b)}{s}, IC^2 = \dfrac{ab(s-c)}{s}$. Hence,

(EIc) $\quad r^2 s = xbc(s-a) + yca(s-b) + zab(s-c) - yza^2 - zxb^2 - xyc^2 \iff$

$xbc(s-a) + yca(s-b) + zab(s-c) - yza^2 - zxb^2 - xyc^2 = (s-a)(s-b)(s-c)$

is equation of incircle.

More applications to inequalities.

For further we will use compact notations for R_a, R_b, R_c for AP, BP, CP respectively.

Application1.

For triangle $\triangle ABC$ with sides a, b, c and arbitrary interior point P holds inequalities:

$\dfrac{a^2+b^2+c^2}{3} \le R_a^2 + R_b^2 + R_c^2$

Proof.

Applying Lagrange's formula to the point $G\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$(medians intersection point) and point P, we obtain

$$PG^2 = \frac{1}{3}\left(PA^2 - GA^2\right) + \frac{1}{3}\left(PB^2 - GB^2\right) + \frac{1}{3}\left(PC^2 - GC^2\right) =$$

$$\frac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \frac{1}{3}\cdot\frac{4}{9}\left(m_a^2 + m_b^2 + m_c^2\right) = \frac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \frac{4}{27}\cdot\frac{3}{4}\left(a^2 + b^2 + c^2\right).$$

Hence, $PG^2 = \dfrac{1}{3}\left(R_a^2 + R_b^2 + R_c^2\right) - \dfrac{a^2+b^2+c^2}{9}$ and that implies inequality

$$\boxed{R_a^2 + R_b^2 + R_c^2 \ge \frac{a^2+b^2+c^2}{3}}$$

.

with equality condition $P = G$ (centroid-medians intersection point).

Application2.

Let x, y, z be any real numbers such that $x + y + z = 1$ and, which can be taken as barycentric coordinates of some point P on plane, that is $(p_a, p_b, p_c) = (x, y, z)$.

Then $\sum_{cyc} xOA^2 - \sum_{cyc} yza^2 = OP^2 \ge 0$ yields inequality

$$\textbf{(R)} \qquad \sum_{cyc} xR_a^2 \ge \sum_{cyc} yza^2,$$

where $R_a := OA, R_b := OB, R_c := OC$ and O is any point in the triangle $T(a, b, c)$.

In homogeneous form this inequality becomes

$$\textbf{(Rh)} \qquad \sum_{cyc} x \cdot \sum_{cyc} xR_a^2 \ge \sum_{cyc} yza^2$$

which holds for any real x, y, z.

If $x := w - v, y := u - w, z := v - u$ then $\sum\limits_{cycc} x = 0$ and we obtain $0 \geq \sum\limits_{cyc} (u - w)(v - u) a^2 \iff$

$$\sum_{cyc} a^2 (u - w)(u - v) \geq 0 \text{ (Schure kind Inequality).}$$

By replacing (x, y, z) in **(R)** with $\left(\frac{x}{R_a^2}, \frac{y}{R_b^2}, \frac{z}{R_c^2}\right)$ we obtain $\sum\limits_{cyc} \frac{x}{R_a^2} \cdot \sum\limits_{cyclic} \frac{x}{R_a^2} \cdot R_a^2 \geq \sum\limits_{cyc} \frac{y}{R_b^2} \cdot \frac{z}{R_c^2} a^2 \iff$

(RR) $\quad \sum\limits_{cyc} x R_b^2 R_c^2 \cdot \sum\limits_{cyclic} x \geq \sum\limits_{cyc} yza^2 R_a^2.$

By substitution $x = aR_a,\ y = bR_b,\ z = cR_c$ in (*) we obtain $\sum\limits_{cycl} aR_a R_b^2 R_c^2 \cdot \sum\limits_{cyc} aR_a \geq \sum\limits_{cyc} bR_b cR_c a^2 R_a^2 \iff \quad \sum\limits_{cyc} aR_b R_c \cdot \sum\limits_{cyc} aR_a \geq abc \cdot aR_a \iff$

(H) $\sum\limits_{cyc} aR_b R_c \geq abc$ (**T.Hayashi inequality**).

08.06.18 To be continued....

Footnote:

Sign ★ before a problem means that this problem is proposed by author of these notes.